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The solution of the Schrödinger equation for complex Hamiltonian systems in two dimensions

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Abstract

We investigate the ground state solutions of the Schrödinger equation for complex (non-Hermitian) Hamiltonian systems in two dimensions within the framework of an extended complex phase-space approach. The eigenvalues and eigenfunctions of some two-dimensional complex potentials are found.

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1. Introduction

In the last few years, very interesting investigations on the PT-symmetric quantum mechanics generated a renewed interest in the analysis of complex potentials [1–5]. These studies show that a complex (non-Hermitian) Hamiltonian can give real and bounded eigenvalues if Hamiltonian is invariant under the simultaneous action of space (P) and time (T) reversal. Therefore, now it is possible to investigate a number of new Hamiltonian systems imposing the PT-symmetric condition [1].

Complex potentials is used to study a variety of phenomena in different fields of physics and chemistry. For example, non-Hermitian Hamiltonians are used in the context of the optical model of a nucleus, to study delocalization transitions in condensed matter systems such as a vortex flux line depinning in type-II superconductors, to study population biology, in the description of a Bose system of hard spheres, to study the energy spectra of complex Toda lattice, quantum cosmology, quantum field theory, supersymmetric quantum mechanics etc [1, 4].

Recently, Kaushal *et al* [5] used the extended complex phase-space approach [6] for solving the Schrödinger equation (SE) for ground and excited states for a number of complex systems in one dimension and discussed the issues related to normalization of the eigenfunctions for non-Hermitian operators. However, their study is restricted to only one-dimensional systems and demands its generalization in higher dimensions. Such extensions in higher dimensions are generally desirable and open new possibilities of studying some more complicated systems. Recently, the extended phase-space approach has also been utilized for tracing complex dynamical invariants (constants of motion) of a number of one-dimensional classical systems [6, 7].

With this motivation, in the present study, we generalize the extended complex phase-space approach in two dimensions with a view to solve the SE for a variety of two-dimensional complex potentials. Although there are various ways of complexifying [6] a given Hamiltonian, but here we use the scheme due to Xavier and de Aguir [8], used to develop an algorithm for the computation of the semiclassical coherent state propagator, to transform potentials on an extended complex phase space.

The organization of the paper is as follows. In section 2, we will develop the extended complex phase-space approach in two dimensions, which enables one to compute eigenvalue spectra of two-dimensional complex systems. In section 3, the eigenvalues and eigenfunctions of some interesting two-dimensional complex systems will be investigated. Finally, concluding remarks are given in section 4.

2. The method

For a two-dimensional complex system described by $H(x, y, p_x, p_y)$, we define the following transformations for the position and momenta variables:

$$\begin{aligned} x &= x_1 + \mathrm{i}p_3, & y &= x_2 + \mathrm{i}p_4, \\ p_x &= p_1 + \mathrm{i}x_3, & p_y &= p_2 + \mathrm{i}x_4. \end{aligned} \tag{1}$$

The presence of variables (x_3, x_4, p_3, p_4) in the above transformations may be regarded as some sort of coordinate–momentum interactions of the dynamical system. Also for the dimensional consideration there appears a constant d in equation (1) in the form $x = x_1 + \mathrm{i}dp_3$, $p_x = p_1 + \mathrm{i}d^{-1}x_3$, etc. In the present work, however, we shall choose $d = 1$ for simplicity. Note that in this complexifying scheme the degrees of freedom of the underlying system just become double and (x_1, p_1), (x_2, p_2), (x_3, p_3) and (x_4, p_4) turn to be canonical pairs.

Similar transformations to equation (1) have also been used in the study of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma [9].

Now, consider the SE (for $\hbar = m = 1$) for two-dimensional systems

$$\hat{H}(x, y, p_x, p_y)\psi(x, y) = E\psi(x, y), \tag{2}$$

where

$$\hat{H}(x, y, p_x, p_y) = -\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right) + V(x, y). \tag{3}$$

Here we only present time-independent stationary state solutions of equation (2) for the sake of convenience. For this purpose, using transformation equation (1), we derive

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial p_3}\right), & \frac{\mathrm{d}}{\mathrm{d}y} &= \frac{1}{2}\left(\frac{\partial}{\partial x_2} - \mathrm{i}\frac{\partial}{\partial p_4}\right), \\ \frac{\mathrm{d}}{\mathrm{d}p_x} &= \frac{1}{2}\left(\frac{\partial}{\partial p_1} - \mathrm{i}\frac{\partial}{\partial x_3}\right), & \frac{\mathrm{d}}{\mathrm{d}p_y} &= \frac{1}{2}\left(\frac{\partial}{\partial p_2} - \mathrm{i}\frac{\partial}{\partial x_4}\right). \end{aligned} \tag{4}$$

Note that the momentum operators $p_x = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ and $p_y = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}y}$ of the conventional quantum mechanics under the transformation (1) reduce to the forms $p_1 + \mathrm{i}x_3 = \frac{-\mathrm{i}}{2}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial p_3}\right)$ and

$p_2 + \mathrm{i}x_4 = \frac{-\mathrm{i}}{2}\left(\frac{\partial}{\partial x_2} - \mathrm{i}\frac{\partial}{\partial p_4}\right)$. These relations give $p_1 = \frac{-1}{2}\frac{\partial}{\partial p_3}, x_3 = \frac{-1}{2}\frac{\partial}{\partial x_1}, p_2 = \frac{-1}{2}\frac{\partial}{\partial p_4}$ and $x_4 = \frac{-1}{2}\frac{\partial}{\partial x_2}$. Thus, these results lead to the commutation relations namely, $[x_1, x_3] = [p_3, p_1] = [x_2, x_4] = [p_4, p_2] = 1, [x_i, p_j] = 0$, where $i, j = 1, 2, 3, 4$.

Also the complex coordinate transformation (1) preserves the fundamental commutation relations, $[x, p_x] = [y, p_y] = i$, which can easily be verified using equations (1) and (4).

Now consider $V(x, y)$, $\psi(x, y)$ and E as complex quantities

$$V(x, y) = V_r(x_1, p_3, x_2, p_4) + \mathrm{i}V_i(x_1, p_3, x_2, p_4),$$
$$\psi(x, y) = \psi_r(x_1, p_3, x_2, p_4) + \mathrm{i}\psi_i(x_1, p_3, x_2, p_4), \qquad E = E_r + \mathrm{i}E_i,$$

where subscripts r and i denote the real and imaginary parts of the corresponding quantities and other subscripts to these quantities separated by comma will denote the partial derivatives of the quantity concerned.

Thus, using equation (4) in equation (3) and using above equations, the SE, equation (2), after separating real and imaginary parts, reduces to a pair of coupled partial differential equations as

$$-\tfrac{1}{8}\left(\psi_{r,x_1x_1} - \psi_{r,p_3p_3} + 2\psi_{i,x_1p_3} + \psi_{r,x_2x_2} - \psi_{r,p_4p_4} + 2\psi_{i,x_2p_4}\right) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \tag{5a}$$

$$-\tfrac{1}{8}\left(\psi_{i,x_1x_1} - \psi_{i,p_3p_3} - 2\psi_{r,x_1p_3} + \psi_{i,x_2x_2} - \psi_{i,p_4p_4} - 2\psi_{r,x_2p_4}\right) + V_r\psi_i + V_i\psi_r = E_r\psi_i + E_i\psi_r. \tag{5b}$$

The Cauchy–Riemann analyticity conditions for $\psi(x, y)$ are given by

$$\begin{aligned} \psi_{r,x_1} &= \psi_{i,p_3}; \qquad & \psi_{r,p_3} &= -\psi_{i,x_1}, \\ \psi_{r,x_2} &= \psi_{i,p_4}; \qquad & \psi_{r,p_4} &= -\psi_{i,x_2}. \end{aligned} \tag{6}$$

Hence, in view of equation (6), we obtain somewhat simpler forms of equations (5a) and (5b) that are written as

$$-\tfrac{1}{2}\left(\psi_{r,x_1x_1} + \psi_{r,x_2x_2}\right) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \tag{7a}$$

$$-\tfrac{1}{2}\left(\psi_{i,x_1x_1} + \psi_{i,x_2x_2}\right) + V_r\psi_i + V_i\psi_r = E_r\psi_i + E_i\psi_r. \tag{7b}$$

Note that equations (7a) and (7b) can be solved for E_r and E_i for a given potential $V(x, y)$ and their explicit forms are given as

$$E_r = -\frac{1}{2\left(\psi_r^2 + \psi_i^2\right)}\left[\psi_r\left(\psi_{r,x_1x_1} + \psi_{r,x_2x_2}\right) + \psi_i\left(\psi_{i,x_1x_1} + \psi_{i,x_2x_2}\right)\right] + V_r, \tag{8a}$$

$$E_i = -\frac{1}{2\psi_r^2 + \psi_i^2}\left[\psi_r\left(\psi_{i,x_1x_1} + \psi_{i,x_2x_2}\right) - \psi_i\left(\psi_{r,x_1x_1} + \psi_{r,x_2x_2}\right)\right] + V_i. \tag{8b}$$

Thus, in order to find solutions of equations (7a) and (7b) (also of equations (8a) and (8b)), we make an ansatz for the eigenfunction $\psi(x, y)$ as

$$\psi(x_1, p_3, x_2, p_4) = \psi_r + \mathrm{i}\psi_i = \exp[g(x, y)] = \exp[g_r(x, y) + \mathrm{i}g_i(x, y)]. \tag{9}$$

The above equation can also be written as

$$\psi_r(x_1, p_3, x_2, p_4) = \mathrm{e}^{g_r}\cos[g_i(x_1, p_3, x_2, p_4)], \tag{10a}$$

$$\psi_i(x_1, p_3, x_2, p_4) = \mathrm{e}^{g_r}\sin[g_i(x_1, p_3, x_2, p_4)]. \tag{10b}$$

Therefore, the analyticity condition, equation (6), for the functions g_r and g_i becomes

$$\begin{aligned} g_{r,x_1} &= g_{i,p_3}; \qquad g_{i,x_1} = -g_{r,p_3}, \\ g_{r,x_2} &= g_{i,p_4}; \qquad g_{i,x_2} = -g_{r,p_4}. \end{aligned} \tag{11}$$

Thus, equations (7*a*) and (7*b*), after using the forms of ψ_r and ψ_i from equations (10*a*) and (10*b*), become

$$g_{r,x_1x_1} + g_{r,x_2x_2} + (g_{r,x_1})^2 + (g_{r,x_2})^2 - (g_{i,x_1})^2 - (g_{i,x_2})^2 + 2(E_r - V_r) = 0, \tag{12a}$$

$$g_{i,x_1x_1} + g_{i,x_2x_2} + 2g_{r,x_1}g_{i,x_1} + 2g_{r,x_2}g_{i,x_2} + 2(E_i - V_i) = 0. \tag{12b}$$

Equations (12*a*) and (12*b*) are now can be rationalized in order to obtain eigenvalue and eigenfunction for a given form of potential. In what follows, we will use the derivations made in the above section to solve SE, equation (2), for a number of two-dimensional complex potentials.

3. Examples

In this section, we consider four two-dimensional complex potentials and solve SE for these cases.

Case 1. First consider a two-dimensional complex potential of the type

$$V(x, y) = ax + by + cx^2 + dy^2 + exy, \tag{13}$$

where the parameters a, b, c, d and e are complex constants.

The real and imaginary parts of the potential, equation (13), using transformation equation (1), are written as

$$\begin{aligned} V_r &= a_r x_1 - a_i p_3 + b_r x_2 - b_i p_4 + c_r(x_1^2 - p_3^2) - 2c_i x_1 p_3 + d_r(x_2^2 - p_4^2) \\ &\quad - 2d_i x_2 p_4 + e_r(x_1x_2 - p_3p_4) - e_i(x_1p_4 + x_2p_3), \end{aligned} \tag{14a}$$

$$\begin{aligned} V_i &= a_i x_1 + a_r p_3 + b_i x_2 + b_r p_4 + c_i(x_1^2 - p_3^2) + 2c_r x_1 p_3 + d_i(x_2^2 - p_4^2) \\ &\quad + 2d_r x_2 p_4 + e_i(x_1x_2 - p_3p_4) + e_r(x_1p_4 + x_2p_3). \end{aligned} \tag{14b}$$

The ansatz for g_r and g_i for the underlying system, which conform equation (11), is considered as

$$\begin{aligned} g_r &= \delta_1 x_1 - \delta_2 p_3 + \delta_3 x_2 - \delta_4 p_4 + \tfrac{1}{2}\alpha_1(x_1^2 - p_3^2) + \tfrac{1}{2}\alpha_2(x_2^2 - p_4^2) \\ &\quad + \beta_1 x_1 p_3 + \beta_2 x_2 p_4 + \gamma_1(x_1x_2 - p_3p_4) - \gamma_2(x_1p_4 + x_2p_3), \end{aligned} \tag{15a}$$

$$\begin{aligned} g_i &= \delta_2 x_1 + \delta_1 p_3 + \delta_4 x_2 + \delta_3 p_4 - \tfrac{1}{2}\beta_1(x_1^2 - p_3^2) - \tfrac{1}{2}\beta_2(x_2^2 - p_4^2) \\ &\quad + \alpha_1 x_1 p_3 + \alpha_2 x_2 p_4 + \gamma_2(x_1x_2 - p_3p_4) + \gamma_1(x_1p_4 + x_2p_3). \end{aligned} \tag{15b}$$

Therefore, using equations (15*a*) and (15*b*) in equations (12*a*) and (12*b*) and equating the coefficients of x_1, x_2, p_3, p_4 and their various products to zero, we obtain a set of non-repeating equations as

$$E_r = -\tfrac{1}{2}(\delta_1^2 - \delta_2^2 + \delta_3^2 - \delta_4^2 + \alpha_1 + \alpha_2), \tag{16a}$$

$$\gamma_1^2 - \gamma_2^2 + \alpha_1^2 - \beta_1^2 = 2c_r, \tag{16b}$$

$$\gamma_1\gamma_2 - \alpha_1\beta_1 = c_i, \tag{16c}$$

$$\gamma_1^2 - \gamma_2^2 + \alpha_2^2 - \beta_2^2 = 2d_r, \tag{16d}$$

$$\gamma_1\gamma_2 - \alpha_2\beta_2 = d_i, \tag{16e}$$

$$\gamma_1(\alpha_1 + \alpha_2) + \gamma_2(\beta_1 + \beta_2) = e_r, \tag{16f}$$

$$\gamma_2(\alpha_1 + \alpha_2) - \gamma_1(\beta_1 + \beta_2) = e_i, \tag{16g}$$

$$\delta_1\alpha_1 + \delta_2\beta_1 + \delta_3\gamma_1 - \delta_4\gamma_2 = a_r, \tag{16h}$$

$$-\delta_1\beta_1 + \delta_2\alpha_1 + \delta_3\gamma_2 + \delta_4\gamma_1 = a_i, \tag{16i}$$

$$\delta_1\gamma_1 - \delta_2\gamma_2 + \delta_3\alpha_2 + \delta_4\beta_2 = b_r, \tag{16j}$$

$$\delta_1\gamma_2 + \delta_2\gamma_1 - \delta_3\beta_2 + \delta_4\alpha_2 = b_i, \tag{16k}$$

$$E_i = -\tfrac{1}{2}(2\delta_1\delta_2 + 2\delta_3\delta_4 - \beta_1 - \beta_2). \tag{16l}$$

In order to find eigenvalues and the corresponding eigenfunction for the system, one should find the solutions of equations (16*b*)–(16*k*).

As such the solutions of various parameters α's, β's, γ's, δ's in the above equations are seem to be difficult. Therefore, in order to seek solutions of these parameters, in terms of the potential coupling parameters (i.e. a, b, c, d, etc), one can make more than one choices among α's, β's, γ's, δ's and can obtain mathematically correct results for E_r, E_i and ψ for a given potential. However, if such general solutions are reduced for some known systems (say, simple harmonic oscillator), these may not provide the well-established results of such systems. Therefore one should make some plausible choices among α's, β's, γ's, δ's while solving equations (16*b*)–(16*k*) in order to avoid any conflict between the general solutions and the well-established results.

Hence keeping such possibilities in mind, we choose $\gamma_1 = \gamma_2$ and $\gamma_1\gamma_2 = -\alpha_1\beta_1$. Thus, for these choices, equations (16*b*)–(16*e*) immediately lead to

$$\alpha_1 = -c_+, \tag{17a}$$

$$\beta_1 = c_-, \tag{17b}$$

$$\alpha_2 = -d_+, \tag{17c}$$

$$\beta_2 = d_-, \tag{17d}$$

where $c_+ = \sqrt{c_r + \sqrt{c_r^2 + c_i^2/4}}$, $c_- = \sqrt{-c_r + \sqrt{c_r^2 + c_i^2/4}}$, $d_+ = \sqrt{d_r + \sqrt{d_r^2 + d_i^2/4 - c_i d_i}}$, and $d_- = \sqrt{-d_r + \sqrt{d_r^2 + d_i^2/4 - c_i d_i}}$ are used.

Further, equations (16*f*) and (16*g*) give two constraining relations on the choices of the potential coupling parameters a, b, c, etc and given as

$$\sqrt{c_i/2}(c_- + d_- - c_+ - d_+) - e_r = 0, \tag{18a}$$

$$\sqrt{c_i/2}(c_- + d_- + c_+ + d_+) + e_i = 0. \tag{18b}$$

Now in order to obtain the solutions for δ_i, $i = 1, 2, 3, 4$, we choose, for simplicity, $\delta_1 = -\delta_3$ and $\delta_2 = -\delta_4$ and utilizing equations (16*h*) and (16*i*) we get

$$\delta_3 = -\delta_1 = [a_r(\sqrt{c_i/2} + c_+) + a_i(\sqrt{c_i/2} + c_-)]/c_1, \tag{19a}$$

$$\delta_2 = -\delta_4 = [a_i(\sqrt{c_i/2} + c_+) - a_r(\sqrt{c_i/2} + c_-)]/c_1, \tag{19b}$$

where $c_1 = \sqrt{4c_r^2 + c_i^2} + c_i + (c_- + c_+)\sqrt{2c_i}$.

Again, equations (16*j*) and (16*k*) provide the following constraining relations:

$$b_r c_1 - (\sqrt{c_i/2} + d_-)[a_i(\sqrt{c_i/2} + c_+) - a_r(\sqrt{c_i/2} + c_-)] + (\sqrt{c_i/2} + d_+)[a_r(\sqrt{c_i/2} + c_+) + a_i(\sqrt{c_i/2} + c_-)] = 0, \quad (20)$$

$$b_i c_1 + (\sqrt{c_i/2} + d_-)[a_r(\sqrt{c_i/2} + c_+) + a_i(\sqrt{c_i/2} + c_-)] + (\sqrt{c_i/2} + d_+)[a_i(\sqrt{c_i/2} + c_+) - a_r(\sqrt{c_i/2} + c_-)] = 0. \quad (21)$$

Thus, after substituting the solutions of $\alpha_i, \beta_i, i = 1, 2,$ and $\delta_j, j = 1, 2, 3, 4$ from equations (17*a*)–(17*d*) and (19*a*), (19*b*) in equations (16*a*) and (16*l*), we find the real and imaginary components of eigenvalue as

$$E_r = (c_+ + d_+)/2 + \left[\left(a_i^2 - a_r^2\right)(2c_r + (c_+ - c_-)\sqrt{2c_i}) - 2a_i a_r(2c_i + (c_- + c_+)\sqrt{2c_i})\right]/c_1^2, \quad (22a)$$

$$E_i = (c_- + d_-)/2 + \left[-4a_i a_r(c_r + (c_+ - c_-)\sqrt{c_i/2}) + 2\left(a_r^2 - a_i^2\right)(c_i + (c_+ + c_-)\sqrt{c_i/2})\right]/c_1^2. \quad (22b)$$

Finally, the eigenfunction is given by

$$\psi = \exp([(1 - \mathrm{i})\sqrt{c_i/2} + c_+ - \mathrm{i}c_-](a_r + \mathrm{i}a_i)(x_2 - x_1 + \mathrm{i}(p_4 - p_3))/c_1 - (c_+ + \mathrm{i}c_-)(x_1 + \mathrm{i}p_3)^2/2 - (d_+ + \mathrm{i}d_-)(x_2 + \mathrm{i}p_4)^2/2 + \sqrt{c_i/2}(1 + \mathrm{i})(x_1 + \mathrm{i}p_3)(x_2 + \mathrm{i}p_4)). \quad (23)$$

Here we wish to mention that the ground state energy and the corresponding eigenfunction of a real two-dimensional harmonic oscillator can easily be obtained by choosing $a = b = e = c_i = d_i = 0$ and $c_r = d_r$, in the potential, equations (13). Hence from equations (22*a*)–(23), we find the expressions for the eigenvalue and eigenfunction as $E_r = \sqrt{2c_r}$, $E_i = 0$, and $\psi = \exp(-\sqrt{c_r/2}(x^2 + y^2))$, which are identical to the results obtained by other methods.

Further, on setting $a_r = b_r = c_i = d_i = e_i = 0$ in equation (13), we obtain the PT-symmetric form of the potential, whose eigenvalue spectra can be obtained using the same ansatz for g_r and g_i as in the present case.

Case 2. Now we consider a two-dimensional harmonic plus inverse harmonic-type potential as

$$V(x, y) = ax^2 + by^2 + cxy + \frac{d}{x^2} + \frac{e}{y^2}, \quad (24)$$

where the parameters a, b, c, d and e are again chosen complex constants. Now using equation (1) in equation (24), we obtain

$$V_r = a_r\left(x_1^2 - p_3^2\right) - 2a_i x_1 p_3 + b_r\left(x_2^2 - p_4^2\right) - 2b_i x_2 p_4 + c_r(x_1 x_2 - p_3 p_4) - c_i(x_1 p_4 + x_2 p_3) + \frac{d_r\left(x_1^2 - p_3^2\right) + 2d_i x_1 p_3}{\left(x_1^2 + p_3^2\right)^2} + \frac{e_r\left(x_2^2 - p_4^2\right) + 2e_i x_2 p_4}{\left(x_2^2 + p_4^2\right)^2}, \quad (25a)$$

$$V_i = a_i\left(x_1^2 - p_3^2\right) + 2a_r x_1 p_3 + b_i\left(x_2^2 - p_4^2\right) + 2b_r x_2 p_4 + c_i(x_1 x_2 - p_3 p_4) + c_r(x_1 p_4 + x_2 p_3) + \frac{d_i\left(x_1^2 - p_3^2\right) - 2d_r x_1 p_3}{\left(x_1^2 + p_3^2\right)^2} + \frac{e_i\left(x_2^2 - p_4^2\right) - 2e_r x_2 p_4}{\left(x_2^2 + p_4^2\right)^2}, \quad (25b)$$

after separating real and imaginary parts. Here we make an ansatz for the eigenfunction for this potential, which is consistent to condition (11), as

$$g_r = \frac{\alpha_1}{2}\left(x_1^2 - p_3^2\right) + \frac{\alpha_2}{2}\left(x_2^2 - p_4^2\right) + \beta_1 x_1 p_3 + \beta_2 x_2 p_4 + \gamma_1(x_1x_2 - p_3p_4) - \gamma_2(x_1p_4 + x_2p_3)$$
$$+\delta_1 \tan^{-1}\frac{x_1}{p_3} + \delta_2 \tan^{-1}\frac{x_2}{p_4} - \frac{\delta_3}{2}\ln\left(x_1^2 + p_3^2\right) - \frac{\delta_4}{2}\ln\left(x_2^2 + p_4^2\right), \tag{26a}$$

$$g_i = \frac{-\beta_1}{2}\left(x_1^2 - p_3^2\right) - \frac{\beta_2}{2}\left(x_2^2 - p_4^2\right) + \alpha_1 x_1 p_3 + \alpha_2 x_2 p_4 + \gamma_2(x_1x_2 - p_3p_4) + \gamma_1(x_1p_4 + x_2p_3)$$
$$+\delta_3 \tan^{-1}\frac{x_1}{p_3} + \delta_4 \tan^{-1}\frac{x_2}{p_4} + \frac{\delta_1}{2}\ln\left(x_1^2 + p_3^2\right) + \frac{\delta_2}{2}\ln\left(x_2^2 + p_4^2\right). \tag{26b}$$

The rationalization of equations (12a) and (12b), after using equations (26a) and (26b), produce a set of equations as

$$E_r = -\tfrac{1}{2}(\alpha_1 + \alpha_2 + 2(\delta_1\beta_1 + \delta_2\beta_2 - \delta_3\alpha_1 - \delta_4\alpha_2)), \tag{27a}$$
$$\gamma_1^2 - \gamma_2^2 + \alpha_1^2 - \beta_1^2 = 2a_r, \tag{27b}$$
$$\gamma_1\gamma_2 - \alpha_1\beta_1 = a_i, \tag{27c}$$
$$\gamma_1^2 - \gamma_2^2 + \alpha_2^2 - \beta_2^2 = 2b_r, \tag{27d}$$
$$\gamma_1\gamma_2 - \alpha_2\beta_2 = b_i, \tag{27e}$$
$$\gamma_1(\alpha_1 + \alpha_2) + \gamma_2(\beta_1 + \beta_2) = c_r, \tag{27f}$$
$$\gamma_2(\alpha_1 + \alpha_2) - \gamma_1(\beta_1 + \beta_2) = c_i, \tag{27g}$$
$$-\delta_1 - 2\delta_1\delta_3 = 2d_i, \tag{27h}$$
$$\delta_3 + \delta_3^2 - \delta_1^2 = 2d_r, \tag{27i}$$
$$-\delta_2 - 2\delta_2\delta_4 = 2e_i, \tag{27j}$$
$$\delta_4 + \delta_4^2 - \delta_2^2 = 2e_r, \tag{27k}$$
$$\delta_1\gamma_1 - \delta_3\gamma_2 = 0, \tag{27l}$$
$$\delta_1\gamma_2 + \delta_3\gamma_1 = 0, \tag{27m}$$
$$\delta_2\gamma_1 - \delta_4\gamma_2 = 0, \tag{27n}$$
$$\delta_2\gamma_2 + \delta_4\gamma_1 = 0, \tag{27o}$$
$$E_i = \tfrac{1}{2}(\beta_1 + \beta_2 - 2(\delta_1\alpha_1 + \delta_2\alpha_2 + \delta_3\beta_1 + \delta_4\beta_2)). \tag{27p}$$

Again to derive the solutions for various parameters in the above equations, we assume $\gamma_1 = \gamma_2$ and $\gamma_1\gamma_2 = -\alpha_1\beta_1$. Thus from equations (27b)–(27e), we find

$$\gamma_1 = \gamma_2 = \sqrt{\frac{a_i}{2}}, \tag{28a}$$
$$\alpha_1 = -a_+, \qquad \beta_1 = a_-, \tag{28b}$$
$$\alpha_2 = -b_+, \qquad \beta_2 = b_-, \tag{28c}$$

where $a_+ = \left(a_r + \left(a_r^2 + \frac{a_i^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$, $a_- = \left(-a_r + \left(a_r^2 + \frac{a_i^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$, $b_+ = \left(b_r + \left(|b|^2 - a_ib_i + \frac{a_i^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$ and $b_- = \left(-b_r + \left(|b|^2 - a_ib_i + \frac{a_i^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$ with $|b| = \left(b_r^2 + b_i^2\right)^{\frac{1}{2}}$.

Equations (27*f*) and (27*g*) provide two constraining relations among the potential coupling parameters and written as

$$(a_+ - a_- + b_+ - b_-)\sqrt{a_i/2} + c_r = 0, \tag{29a}$$

$$(a_+ + a_- + b_+ + b_-)\sqrt{a_i/2} + c_i = 0. \tag{29b}$$

Further, using equations (27*h*)–(27*k*), we obtain the solutions for the remaining parameters δ_i's as

$$\delta_1 = \mp \frac{4d_i}{\sqrt{2 + 16d_r \pm 2\sqrt{1 + 16(d_r + 4|d|^2)}}}, \tag{30a}$$

$$\delta_2 = \mp \frac{4e_i}{\sqrt{2 + 16e_r \pm 2\sqrt{1 + 16(e_r + 4|e|^2)}}}, \tag{30b}$$

$$\delta_3 = -\frac{1}{2} \pm \frac{1}{4}\sqrt{2 + 16d_r \pm 2\sqrt{1 + 16(d_r + 4|d|^2)}}, \tag{30c}$$

$$\delta_4 = -\frac{1}{2} \pm \frac{1}{4}\sqrt{2 + 16e_r \pm 2\sqrt{1 + 16(e_r + 4|e|^2)}}. \tag{30d}$$

Note that equations (27*l*)–(27*o*) give four more constraining relations.

Finally, the eigenvalues and the eigenfunction may be obtained from equations (27*a*) and (27*p*), and equations (26*a*) and (26*b*) respectively, after substituting the values of α's, β's and δ's, as

$$\begin{aligned} E_r = &\pm \frac{4d_i a_-}{\sqrt{2 + 16d_r \pm 2\sqrt{1 + 16(d_r + 4|d|^2)}}} \pm \frac{4e_i b_-}{\sqrt{2 + 16e_r \pm 2\sqrt{1 + 16(e_r + 4|e|^2)}}} \\ &- a_+ \left(-\frac{1}{2} \pm \frac{1}{4}\sqrt{2 + 16d_r \pm 2\sqrt{1 + 16(d_r + 4|d|^2)}}\right) \\ &- b_+ \left(-\frac{1}{2} \pm \frac{1}{4}\sqrt{2 + 16e_r \pm 2\sqrt{1 + 16(e_r + 4|e|^2)}}\right) - \frac{1}{2}(a_+ + b_+), \end{aligned} \tag{31a}$$

$$\begin{aligned} E_i = &\pm \frac{4d_i a_+}{\sqrt{2 + 16d_r \pm 2\sqrt{1 + 16(d_r + 4|d|^2)}}} \pm \frac{4e_i b_+}{\sqrt{2 + 16e_r \pm 2\sqrt{1 + 16(e_r + 4|e|^2)}}} \\ &+ a_- \left(-\frac{1}{2} \pm \frac{1}{4}\sqrt{2 + 16d_r \pm 2\sqrt{1 + 16(d_r + 4|d|^2)}}\right) \\ &+ b_- \left(-\frac{1}{2} \pm \frac{1}{4}\sqrt{2 + 16e_r \pm 2\sqrt{1 + 16(e_r + 4|e|^2)}}\right) + \frac{1}{2}(a_- + b_-), \end{aligned} \tag{31b}$$

and

$$\begin{aligned} \psi = &\left(x_1^2 + p_3^2\right)^{\frac{i}{2}(\delta_1 + i\delta_3)} \left(x_2^2 + p_4^2\right)^{\frac{i}{2}(\delta_2 + i\delta_4)} \exp\left[-\frac{1}{2}(a_+ + ia_-)(x_1 + ip_3)^2\right. \\ &- \frac{1}{2}(b_+ + ib_-)(x_2 + ip_4)^2 + (1 + i)\sqrt{a_i/2}(x_1 + ip_3)(x_2 + ip_4) \\ &\left. + (\delta_1 + i\delta_2)\tan^{-1}\frac{x_1}{p_3} + (\delta_2 + i\delta_4)\tan^{-1}\frac{x_2}{p_4}\right]. \end{aligned} \tag{32}$$

Similarly one can also find the explicit forms of eigenvalues and eigenfunction of the PT-symmetric version of the potential (24) using the same ansatz, equations (26*a*) and (26*b*).

Case 3. Here, we consider a potential of the type

$$V(x, y) = ax^2 + by^2 + cxy + \frac{dx}{y} + \frac{ey}{x}, \tag{33}$$

with a, b, c, d and e as complex quantities. Integrability of some particular forms of this potential has been studied in past [10]. For this potential, we assume g_r and g_i as

$$\begin{aligned} g_r = {} & \frac{\alpha_1}{2}\left(x_1^2 - p_3^2\right) + \frac{\alpha_2}{2}\left(x_2^2 - p_4^2\right) + \beta_1 x_1 p_3 + \beta_2 x_2 p_4 + \eta_1(x_1 x_2 - p_3 p_4) \\ & - \eta_2(x_1 p_4 + x_2 p_3) + \delta_1 \tan^{-1}\frac{x_1}{p_3} + \delta_2 \tan^{-1}\frac{x_2}{p_4} \\ & - \frac{1}{2}\delta_3 \ln\left(x_1^2 + p_3^2\right) - \frac{1}{2}\delta_4 \ln\left(x_2^2 + p_4^2\right), \end{aligned} \tag{34a}$$

$$\begin{aligned} g_i = {} & \frac{-\beta_1}{2}\left(x_1^2 - p_3^2\right) - \frac{\beta_2}{2}\left(x_2^2 - p_4^2\right) + \alpha_1 x_1 p_3 + \alpha_2 x_2 p_4 + \eta_2(x_1 x_2 - p_3 p_4) \\ & + \eta_1(x_1 p_4 + x_2 p_3) + \delta_3 \tan^{-1}\frac{p_3}{x_1} + \delta_4 \tan^{-1}\frac{p_4}{x_2} \\ & + \frac{1}{2}\delta_1 \ln\left(x_1^2 + p_3^2\right) + \frac{1}{2}\delta_2 \ln\left(x_2^2 + p_4^2\right). \end{aligned} \tag{34b}$$

Again we find a set of equations among the parameters of g_r and g_i, after using equations (34*a*) and (34*b*) in equations (12*a*) and (12*b*) as

$$E_r = -\tfrac{1}{2}(\alpha_1 + \alpha_2 + 2(\delta_1\beta_1 + \delta_2\beta_2 - \delta_3\alpha_1 - \delta_4\alpha_2)), \tag{35a}$$

$$\eta_1\eta_2 - \alpha_1\beta_1 = a_i, \tag{35b}$$

$$\eta_1^2 - \eta_2^2 + \alpha_1^2 - \beta_1^2 = 2a_r, \tag{35c}$$

$$\eta_1\eta_2 - \alpha_2\beta_2 = b_i, \tag{35d}$$

$$\eta_1^2 - \eta_2^2 + \alpha_2^2 - \beta_2^2 = 2b_r, \tag{35e}$$

$$\eta_1(\alpha_1 + \alpha_2) + \eta_2(\beta_1 + \beta_2) = c_r, \tag{35f}$$

$$\eta_2(\alpha_1 + \alpha_2) - \eta_1(\beta_1 + \beta_2) = c_i, \tag{35g}$$

$$\eta_1\delta_4 + \eta_2\delta_2 = -d_r, \tag{35h}$$

$$\eta_1\delta_2 - \eta_2\delta_4 = d_i, \tag{35i}$$

$$\eta_1\delta_3 + \eta_2\delta_1 = -e_r, \tag{35j}$$

$$\eta_1\delta_1 - \eta_2\delta_3 = e_i, \tag{35k}$$

$$\delta_3 + \delta_3^2 - \delta_1^2 = 0, \tag{35l}$$

$$\delta_1(1 + 2\delta_3) = 0, \tag{35m}$$

$$\delta_4 + \delta_4^2 - \delta_2^2 = 0, \tag{35n}$$

$$\delta_2(1 + 2\delta_4) = 0, \tag{35o}$$

$$E_i = \tfrac{1}{2}(\beta_1 + \beta_2 - 2(\alpha_1\delta_1 + \alpha_2\delta_2 + \beta_1\delta_3 + \beta_2\delta_4)). \tag{35p}$$

Again in order to find the solutions of the above equations, we choose $\eta_1 = \eta_2$ and $\eta_1\eta_2 = -\alpha_1\beta_1$. These choices immediately provide the solutions for α's, β's, η's and δ's. Thus from equations (35*b*)–(35*e*) we obtain

$$\eta_1 = \eta_2 = \sqrt{\frac{a_i}{2}}, \tag{36a}$$

$$\alpha_1 = -a_+, \qquad \beta_1 = a_-, \tag{36b}$$

$$\alpha_2 = -b_+, \qquad \beta_2 = b_-, \tag{36c}$$

where the definitions of a_+, a_-, b_+ and b_- are same as given in case 2. Also equations (35*f*) and (35*g*) produce two constraining relations, which are same as given in equations (29*a*) and (29*b*).

Similarly from equations (35*h*)–(35*k*), we get

$$\delta_1 = \frac{-e_r + e_i}{\sqrt{2a_i}} \qquad \delta_2 = \frac{-d_r + d_i}{\sqrt{2a_i}}, \tag{37a}$$

$$\delta_3 = \frac{-e_r - e_i}{\sqrt{2a_i}} \qquad \delta_4 = \frac{-d_r - d_i}{\sqrt{2a_i}}. \tag{37b}$$

Again one can derive four more constraining expressions by substituting δ's from equations (37*a*) and (37*b*) in equations (35*l*)–(35*o*).

Finally, the eigenvalues are given as

$$E_r = \tfrac{1}{2}(a_+ + b_+) + [(a_+ + a_-)e_r + (a_+ - a_-)e_i + (b_+ + b_-)d_r + (b_+ - b_-)d_i]/\sqrt{2a_i}, \tag{38}$$

$$E_i = \tfrac{1}{2}(a_- + b_-) + [(a_- - a_+)e_r + (a_+ + a_-)e_i + (b_- - b_+)d_r + (b_+ + b_-)d_i]/\sqrt{2a_i}, \tag{39}$$

and the eigenfunction becomes

$$\psi = \left(x_1^2 + p_3^2\right)^{\frac{(e_r+\mathrm{i}e_i)(1-\mathrm{i})}{\sqrt{8a_i}}} \left(x_2^2 + p_4^2\right)^{\frac{(d_r+\mathrm{i}d_i)(1-\mathrm{i})}{\sqrt{8a_i}}} \exp\left[-\frac{1}{2}(a_+ + \mathrm{i}a_-)(x_1 + \mathrm{i}p_3)^2 - \frac{1}{2}(b_+ + \mathrm{i}b_-)(x_2 + \mathrm{i}p_4)^2 + (1+\mathrm{i})\sqrt{a_i/2}(x_1 + \mathrm{i}p_3)(x_2 + \mathrm{i}p_4) - \frac{(e_r + \mathrm{i}e_i)(1-\mathrm{i})}{\sqrt{8a_i}} \tan^{-1}\frac{x_1}{p_3} - \frac{(d_r + \mathrm{i}d_i)(1-\mathrm{i})}{\sqrt{8a_i}} \tan^{-1}\frac{x_2}{p_4}\right]. \tag{40}$$

Case 4. Finally, we consider the PT-symmetric version of the case three potential, equation (33), which is obtained by setting $a_i = b_i = c_i = d_i = e_i = 0$ and is given by

$$V(x, y) = a_r x^2 + b_r y^2 + c_r xy + \frac{d_r x}{y} + \frac{e_r y}{x}. \tag{41}$$

Note that in this potential only real coupling parameters are present.

The eigenvalues and the eigenfunction for this case can be obtained using the same ansatz for g_r and g_i used in case 3. As a result, equations (35*b*)–(35*m*) reduce to some simpler forms. The solution of these equations may be obtained by choosing $\beta_1 = \beta_2 = \eta_2 = 0$ and $\alpha_1 = \eta_1$. Hence the solutions are given as

$$\alpha_1 = \eta_1 = -\sqrt{a_r}, \qquad \alpha_2 = -\sqrt{2b_r - a_r}, \tag{42a}$$

$$\delta_1 = \delta_2 = 0, \qquad \delta_3 = \frac{c_r}{\sqrt{a_r}}, \qquad \delta_4 = \frac{d_r}{\sqrt{a_r}}, \tag{42b}$$

with a restriction $\sqrt{a_r} + \sqrt{2b_r - a_r} - c_r/\sqrt{a_r} = 0$.

Finally, the eigenvalues and the eigenfunction turn out as

$$E_r = \frac{1}{2}(\sqrt{a_r} + \sqrt{2b_r - a_r} - 2c_r - 2d_r\sqrt{2b_r/a_r - 1}), \qquad E_i = 0, \tag{43}$$

$$\psi = \left(x_1^2 + p_3^2\right)^{\frac{-c_r}{2\sqrt{a_r}}} \left(x_2^2 + p_4^2\right)^{\frac{-d_r}{2\sqrt{a_r}}} \exp\left[-\frac{1}{2}\sqrt{a_r}(x_1 + \mathrm{i}p_3)^2 - \frac{1}{2}\sqrt{2b_r - a_r}(x_2 + \mathrm{i}p_4)^2 + \sqrt{a_r}(x_1 + \mathrm{i}p_3)(x_2 + \mathrm{i}p_4) + \frac{\mathrm{i}c_r}{\sqrt{a_r}} \tan^{-1}\frac{x_1}{p_3} + \frac{\mathrm{i}d_r}{\sqrt{a_r}} \tan^{-1}\frac{x_2}{p_4}\right]. \tag{44}$$

Note that the imaginary part of the eigenvalue is zero, which is same as observed for one-dimensional systems [5].

4. Conclusions

In the present work, we have tried to develop the extended phase-space approach [5], for two-dimensional complex Hamiltonian systems. This extension can be utilized to study quantum mechanics of some more realistic two-dimensional complex systems.

Although in our study of one- and two-dimensional complex systems using the present approach, we observed that if the eigenvalue and eigenfunction of a complex system in one dimension is known, then the eigenvalue spectra of an analogous two-dimensional system can straightforwardly be obtained by inserting similar terms, as present in one dimension, for the second coordinate, provided the same form of ansatz for the eigenfunction is used. For example, for a one-dimensional harmonic oscillator system, $H = ax^2$, with complex $a = a_r + \mathrm{i}a_i$, the real and imaginary parts of eigenvalue are $E_r = \sqrt{a_r + |a|}$ and $E_i = \sqrt{-a_r + |a|}$ and the eigenfunction is $\psi(x_1, p_3) = \exp\left[\frac{-(a+|a|)(x_1+\mathrm{i}p_3)^2}{4\sqrt{a_r+|a|}}\right]$ [5] and for its two-dimensional counterpart, $H = ax^2 + by^2, a$ and b are complex, one can obtain $E_r = \sqrt{a_r + |a|} + \sqrt{b_r + |b|}$, $E_i = \sqrt{-a_r + |a|} + \sqrt{-b_r + |b|}$ and $\psi(x_1, p_3, x_2, p_4) = \exp\left[\frac{-(a+|a|)(x_1+\mathrm{i}p_3)^2}{4\sqrt{a_r+|a|}} + \frac{-(b+|b|)(x_2+\mathrm{i}p_4)^2}{4\sqrt{b_r+|b|}}\right]$. However, this generalization works only for separable two-dimensional systems and fails for other two-dimensional systems having interaction/cross terms in the potential function. For such systems, one should select suitable ansatz for the eigenfunction for obtaining eigenvalue spectra.

In view of these observations, in the present study, we considered four two-dimensional complex potentials and found their corresponding eigenvalues and eigenfunctions. In case 4, which is the PT-symmetric version of case 3, the imaginary part of the eigenvalue is zero, i.e. $E_i = 0$ even in the presence of cross terms in the potential. This particular observation again shows that for the existence of nonzero E_i, the potential should have some imaginary coupling terms.

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References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
Bender C M, Dorje C B and Hugh F J 2003 *Am. J. Phys.* **71** 1095
Bender C M, Brod J, Refig A and Reuter M E 2004 *J. Phys. A: Math. Gen.* **37** 10139
[2] Bagchi B and Roy Choudhury R 2000 *J. Phys. A: Math. Gen.* **33** L1
Khare A and Mandal B P 2000 *Phys. Lett.* A **272** 53
[3] Mastafazadeh A 2002 *J. Math. Phys.* **43** 205, 2814, 3944
Ahmed Z 2002 *Phys. Lett.* A **294** 287
[4] Andrrianov A A, Cannata F and Kamenshchik A Y 2006 *J. Phys. A: Math. Gen.* **39** 9975
[5] Kaushal R S 2001 *J. Phys. A: Math. Gen.* **34** L709
Kaushal R S and Parthasarthi 2002 *J. Phys. A: Math. Gen.* **35** 8743
Parthasarthi and Kaushal R S 2003 *Phys. Scr.* **68** 115
[6] Kaushal R S and Singh S 2001 *Ann. Phys., NY* **288** 253
[7] Chand F and Mishra S C 2006 *Pramana J. Phys.* **66** 999
[8] Xavier A L Jr and de Aguiar M A M 1996 *Ann. Phys., NY* **252** 458
Xavier A L Jr and de Aguiar M A M 1997 *Phys. Rev. Lett.* **79** 3323
[9] Rao N N, Buti B and Khadkikar S B 1986 *Pramana J. Phys.* **27** 497
[10] Hietarinta J 1984 *Phys. Rev. Lett.* **52** 1057
Hall L S 1985 *Phys. Rev. Lett.* **54** 614